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Development of a Fuzzy Probabilistic Methodology for Multiple-Site Fatigue Damage

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A strategy for fuzzy-probabilistic assessment of the impact of multiple-site fatigue damage (MSD) on the fatigue resistance of aging structures is developed. The residual strength of a structure may be significantly reduced by the existence of fatigue damage at multiple locations. Depending on the level of knowledge with which they are known, MSD-related parameters may be represented as either purely random variables or fuzzy random variables. The membership functions of the probabilistic characteristics of fuzzy random variables, namely, mean values and standard deviations are developed. Mechanistic and probabilistic models used to evaluate multi-site fatigue damage are also presented. A probabilistic solution strategy, employing a first-order reliability method, is combined with a response-surface-based fuzzy modeling approach to construct the possibility distributions of the probabilistic safety indicators (namely, reliability indices and failure probabilities) for components subjected to multiple-site fatigue damage. Instead of providing the traditional single-valued, purely probabilistic measure for reliability, the present formulation proves its merit in its ability to combine experimental data with expert knowledge to provide confidence bounds on the structural integrity of aging structures. Moreover, the predicted bounds are dependent on the level of knowledge regarding the fuzzy input parameters, with a greater knowledge producing more narrow bounds. An example problem is used to demonstrate the advantages of the proposed methodology.

Introduction

IN-SERVICE aging constitutes an area of major concern for most structures. Because both new and aging structures must typically meet or exceed the same structural integrity requirements, structural longevity must be diligently considered during both the initial design process and while addressing necessary repair measures. Whereas the assessment of structural integrity and reliability is founded on an accurate prediction of in-service damage accumulation, few in the industry recognize the potentially dire consequences associated with the presence of tiny, virtually undetectable fatigue cracks in structures, a phenomenon commonly referred to as widespread-fatigue damage (WFD). Two categories of WFD include multiple-site damage (MSD), which refers to WFD within the same structural element, and multiple-element damage, which denotes extensive fatigue-induced damage within adjacent structural components. In the aviation industry, for example, aircraft structures are currently designed to possess large lead-crack damage tolerances. WFD may significantly compromise this tolerance, and its detection is unfortunately limited by the capabilities of current inspection technologies. As such, minimizing the effects of WFD requires an accurate and reliable estimate as to when WFD-induced degradation of residual strength would become unacceptable and then repairing the structure before such time. Such a formidable task is further complicated by the fact that structures are continuously aging and often operating beyond their original design lives. Enhancing knowledge of multiple-site fatigue damage, through both elicitation of subject matter experts (SMEs) and other viable sources, is of utmost importance to both the improvement of current structural designs and also the development of more effective inspection and maintenance management programs for aging structures.

Durability- and damage-tolerance-based strategies currently employed to evaluate the effects of MSD on structural integrity are based largely on the application of fracture mechanics and fatigue crack growth laws. Such laws constitute the foundation for many inspection scheduling and life-extension decisions. Much experimental testing has been conducted to evaluate structural response in the presence of MSD.[1,2] Such investigations are typically based on virgin material, or at best, artificially aged specimens. However, many of the parameters associated with MSD are largely uncertain, especially with age. As a result, researchers[3,4] have expanded deterministic-based approaches into the probabilistic realm, in which the once deterministic MSD parameters are considered random, each defined by characteristics such as mean, standard deviation, parametric correlation, and distribution type. However, an element of uncertainty may even be inherent in the probabilistic characterization of a set of random variables. Moreover, with age, a structure's original properties, as well as the certainty with which they were once known, tend to degrade somewhat. In some cases, these uncertainties are objective in nature, in which case they can be modeled by probabilistic mechanics. In many instances, however, the probabilistic characteristics (mean, standard deviation, correlation, etc.) of the variables involved are not precisely known, thereby rendering the uncertainty subjective. Such parametric subjectivity is more suitably modeled by means of fuzzy logic methods, which combine experimental data with knowledge gained through elicitation of SMEs to provide ultimately confidence bounds on structural reliability. Both parametric uncertainty and in-service aging of structures may be accounted for through careful fuzzy probabilistic characterization and analysis, in conjunction with expert (SME) elicitation.

Various strategies for modeling uncertainty have been presented in the literature, including concepts involving stochastics (probabilistics), fuzziness, antioptimization, and convex modeling. A thorough review of such modeling strategies is provided in Refs. 5–7. The objective of the current study is to develop a fuzzy probabilistic framework with which the effects of multiple-site fatigue cracking in structures may be modeled. The approach synergistically combines the fuzzy modeling strategy developed by Akpan et al.[8] with probabilistic models for MSD,[3] thereby, providing a methodology for computing the possibility distribution of probabilistic quantities such as reliability index β or failure probability

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P_f for structures under the influence of multiple-site fatigue damage. Traditional purely probabilistic methods yield a single value for β and P_f, which ignores inherent uncertainty in knowledge of the impact of MSD on residual strength and whose confidence bounds cannot be accurately defined. The merit of the approach presented herein lies in that, in lieu of a single-valued (purely probabilistic) β or P_f, a range of possible values may be specified based on the level of knowledge regarding the relevant input parameters. In general, greater confidence in the input parameters corresponds to a more narrow range of possible output values.

This paper will first outline both the mechanistic approaches to evaluating MSD, including limit state function definition and the probabilistic solution methodologies, which may be used to account for the effects of MSD. An overview of the fuzzy modeling techniques will then be provided, including the response surface method. Finally, an example problem is presented demonstrate the capabilities of the proposed methodology.

Probabilistic and Mechanistic Modeling

Mechanical Formulation and Probabilistic Reliability Models for MSD Problems

Damage tolerance and reliability of structures is based on an accurate prediction of in-service fatigue damage accumulation. Several techniques have been proposed in the literature to account for the detrimental effects of multiple-site fatigue damage. With the numerous uncertainties inherent in both the physical process by which MSD progresses and the mathematical models used in its evaluation, it is being increasingly recognized that the phenomenon is best described using a probabilistic framework.

Evaluation of fracture characteristics and fatigue crack growth of multiple cracks may be accomplished by a number of means. For example, consider the linkup of a lead crack with an MSD crack, which occurs at the point where their corresponding plastic zones meet. Progressive linkup with other MSD cracks would continue until such time as the applied stress σ_a exceeds the component's residual strength σ_r, that is, $\sigma_r \leq \sigma_a$. The residual strength of a plate of width W may be approximated by the following[9]:

$$\sigma_r = \left[W - \phi_{i=1}^{n}(2a_i)\right](\sigma_y/W) \tag{1}$$

in which a_i is the half-crack length of a given lead or MSD crack. It can be shown that lead crack residual strength may be significantly reduced by MSD, the extent of which is extremely sensitive to structural configuration.

Alternatively, finite element (FE) technology may be applied to estimate stress intensity factors (SIF) for structural components subject to MSD. For cases involving a large number of MSD cracks, the practicality of an FE approach is limited by its high computational costs. Other numerical approaches, such as the use of complex stress functions to describe the stress distributions surrounding an MSD configuration, may also prove computationally expensive.

A fracture mechanics-based technique known as the compounding method has proven to be the most versatile and computationally efficient technique, well suited to the numerous model solutions typically required by a probabilistic analysis. Although approximate, the SIF solutions obtained via the compounding method are, nonetheless, reasonably accurate. Because the compounding method will be employed herein, details regarding its application are discussed in the following paragraphs.

In traditional linear elastic fracture mechanics, the mode 1 stress intensity factor range ΔK for a center-cracked panel of finite width W is given by

$$\Delta K = \Delta\sigma\sqrt{\pi a}Y \tag{2}$$

in which $\Delta\sigma$ is the applied stress range, a the half-crack length, and Y a geometric correction factor. The compounding method simply implies the use of an effective or compounded geometric correction factor calculated as the product of several correction factors from existing solutions for simple geometries. In other words, in the case of a component subjected to MSD, the effective stress intensity factor range ΔK_{eff} is given by

$$\Delta K_{\text{eff}} = \Delta\sigma\sqrt{\pi a}Y_{\text{eff}} \tag{3}$$

The compounded geometric correction factor Y_{eff} is given by

$$Y_{\text{eff}} = Y_c Y_h Y_n Y_w Y_m \tag{4}$$

where Y_c accounts for corner cracks, Y_h considers cracks originating at a hole, Y_n accounts for the interaction between neighboring holes, Y_w considers a plate having finite width, and Y_m accounts for mutual crack interaction effects.[3] For a through-thickness crack originating at hole i, Y_n is given by

$$Y_{n_i} = \prod_{i \neq j} Y_{ni,j} \tag{5}$$

where $Y_{ni,j}$ is the hole interaction factor for two holes i (origin) and j (neighboring). Similarly, the crack interaction factor for crack i is given by

$$Y_{m_i} = \prod_{i \neq j} Y_{mi,j} \tag{6}$$

where $Y_{mi,j}$ is the interaction factor for two cracks i and j. Experimental studies[2,10] have shown the compounding factors of most significance to be Y_w and Y_m, reducing Eq. (3) to

$$\Delta K_{\text{eff}} = \Delta\sigma\sqrt{\pi a}(Y_w Y_m) \tag{7}$$

For a panel of finite width (W), Y_w is given by the familiar expression

$$Y_w = \sqrt{\sec(\pi a/W)} \tag{8}$$

where a is the half-crack length. The interaction factors for lead and MSD cracks were developed by Kamei and Yokobori.[11]

A number of performance or limit state functions $g_f(X)$ may be defined for the assessment of structural integrity in the presence of MSD. For cases involving instantaneous fracture and fatigue crack propagation, respectively, two such functions may be formulated in terms of 1) exceedance of the corresponding critical crack length a_{cr} at any time t and 2) exceedance of the critical mode 1 fracture toughness K_{IC} of the material.

The compounding method essentially determines an effective crack configuration, A_{eff}, for which the failure likelihood is given by

$$P_f = P\left(g_{f,A_{\text{eff}}} \leq 0\right), \qquad g_{f,A_{\text{eff}}} = \varphi(A_i) \tag{9}$$

where $g_{f,A_{\text{eff}}}$ defines the limit state in terms of φ, which is in turn a function of the various crack sizes and geometric configuration. For the first fatigue crack growth (FCG) criterion defined in the preceding paragraph, the limit state may be specified in terms of damage functions $\xi(a)$ as

$$g_{f,A_{\text{eff}}}(t) = \xi(a_{\text{cr}}) - \xi[a(t)] \tag{10}$$

where a_{cr} is the critical crack length and $a(t)$ the crack length at time t. The damage functions $\xi(a)$ are given by

$$\xi(a) = \int_{a_0}^{a}\left(\frac{\mathrm{d}a}{\left\{\sqrt{\pi a}[Y_{\text{eff}}(a)]\right\}^m}\right) = \cdots = \int_{N_0}^{N} C(\Delta\sigma)^m\,\mathrm{d}N \tag{11}$$

where a_0 is the initial crack length.

Orisamolu[3] proposed an extension of the described approach to cases involving multiple cracks. Consider, for example, the interaction of two cracks i and j, of lengths a_i and a_j, respectively. The corresponding critical damage functions are given by

$$\xi(a_{i,\text{cr}}) = \int_{a_{i_0}}^{a_{i,\text{cr}}}\left(\frac{\mathrm{d}a_i}{\left\{\sqrt{\pi a_i}[Y_{\text{eff}}(a_i, a_j)]\right\}^m}\right) \tag{12}$$

$$\xi(a_{j,\text{cr}}) = \int_{a_{j_0}}^{a_{j,\text{cr}}}\left(\frac{\mathrm{d}a_j}{\left\{\sqrt{\pi a_j}[Y_{\text{eff}}(a_i, a_j)]\right\}^m}\right) \tag{13}$$

whereas the corresponding damage functions at time t are given by

$$\xi[a_i(t)] = \int_{a_{i0}}^{a_i(t)} \left[\frac{\mathrm{d}a_i}{\left\{\sqrt{\pi a_i}[Y_{\mathrm{eff}}(a_i, a_j)]\right\}^m} \right] \tag{14}$$

$$\xi[a_j(t)] = \int_{a_{j0}}^{a_j(t)} \left(\frac{\mathrm{d}a_j}{\left\{\sqrt{\pi a_j}[Y_{\mathrm{eff}}(a_i, a_j)]\right\}^m} \right) \tag{15}$$

Moreover, an effective FCG-based limit state function for multiple interacting cracks may be defined as follows[3]:

$$g_{f,A_{\mathrm{eff}}}(t) = \underset{i=1,2,\ldots,n}{\mathrm{Minimize}} \{\xi(a_{i,\mathrm{cr}}) - \xi[a_i(t)]\} \tag{16}$$

Similarly, the second fracture criterion mentioned earlier may be defined by the following limit state:

$$g_{f,A_{\mathrm{eff}}} = K_{IC} - (\psi_{\Delta K})\Delta K_{A_{\mathrm{eff}}} \tag{17}$$

in which K_{IC} is the mode 1 fracture toughness and $\Psi_{\Delta K}$ a bias correction factor for the estimation of the effective stress intensity factor range for the effective crack configuration[3] ($\Delta K_{A_{\mathrm{eff}}}$).

Probabilistic Solution Strategies

Either of Eqs. (16) or (17) can be used to compute the failure probability of an aging structure in the presence of multiple-site fatigue damage. However, note that, because some of the model parameters are not only random in nature, but also fuzzy to some extent, there are many possible realizations for the probability of failure and reliability index. A strategy for computing the possibility distribution for these probabilistic response quantities will be developed in the next section. The limit state function $g(X)$ is typically defined such that

$$\begin{aligned} g(X) < 0 &\quad \Rightarrow \quad \text{failure} \\ g(X) = 0 &\quad \Rightarrow \quad \text{limit state boundary} \\ g(X) > 0 &\quad \Rightarrow \quad \text{no failure} \end{aligned} \tag{18}$$

The instantaneous reliability of a structure may be calculated based on one of the earlier defined limit states, where the failure domain is defined by $\Omega = [g(X) < 0]$ and its compliment $\Omega^{|} = [g(X) > 0]$ defines the safe domain. The instantaneous failure probability at time t is defined by

$$P_f(t) = \int_{\Omega} f[X(t)]\,\mathrm{d}x \tag{19}$$

where $f[X(t)]$ is the joint probability density function of the basic random variables at time t. In general, the joint probability density function is unknown, and evaluating the convolution integral is a formidable task. Several practical approaches have been developed, including first-order reliability method (FORM) and second-order reliability method (SORM).

FORM, also known as fast probability integration scheme, is the most robust methodology for computing instantaneous failure probability. The method uses the Hasofer–Lind formulation, developed in 1974. The basic concept of the Hasofer–Lind formulation [or Advanced First-Order Second Moment (AFOSM) model] involves the transformation of Gaussian, that is, normal, random variables to the standard form (mean $\mu = 0$ and standard deviation $\sigma = 1$. The Hasofer–Lind (HL) reliability index β_{HL} is then computed as the minimum distance from the origin to the limit state surface.

Although the HL formulation is limited to cases involving normally distributed variables, the work represents an important milestone and has laid a solid foundation for the development of a class of procedures generically referred to as FORM. FORM procedures utilize optimization-based techniques to evaluate the reliability index β, from which the failure probability P_f can then be obtained using the relationship:

$$\beta = \Phi^{-1}(P_f) \tag{20}$$

where Φ is the standard normal cumulative distribution function. FORM procedures utilize the full distribution information of the random variables involved in the definition of the limit state function and parametric correlation is permitted. Several techniques are available with which to perform FORM calculations. It is sufficient, however, to illustrate the basic features of the entire class via a description of a particular scheme called the HL–Rackwitz–Fiessler (RF) algorithm. The HL–RF algorithm is named after Hasofer and Lind based on the work described earlier, and Rackwitz and Fiessler, who in 1978 first proposed the generalization of the HL scheme to non-Gaussian random variables. The HL–RF algorithm is one of the most popular FORM procedures.

The essential steps involved in FORM algorithms include 1) transformation of the vector of basic random variables X from the original X space to the standard normal U space; 2) a search (usually in U space) for the point (denoted by U^*) on the limit state surface $g(U) = 0$ that has the highest joint probability density, the point popularly referred to as the design point, failure point, or the most probable point (MPP); 3) an approximation (at the MPP) of the failure surface in U space; and 4) computation of the distance from the origin to the MPP, referred to as reliability index β, from which failure probability follows [Eq. (20)].

The transformation from the original (X space) to the standard normal (U space) is usually denoted by the transformation operator T such that

$$U = T(X) \tag{21}$$

The probability transformation scheme described earlier has been shown to yield extremely accurate results in reliability analysis.

The search referred to in step 2 is conducted by means of the solution of an optimization problem. The optimization problem pertaining to the calculation of the HL reliability index in the U space may be posed as follows:

$$\text{minimize } D = \sqrt{U_i^T U_i} = \beta, \qquad \text{subject to } g(U_i) = 0 \tag{22}$$

The solution of this problem locates the MPP and the n-dimensional position vector defining this point, $\boldsymbol{U}^*$, is given by

$$U^* = \boldsymbol{\alpha}^*\beta \tag{23}$$

where $\boldsymbol{\alpha}^*$ is the unit normal vector at the MPP, which is in turn given by

$$\boldsymbol{\alpha}^* = \frac{\nabla g(\boldsymbol{U}^*)}{|\nabla g(\boldsymbol{U}^*)|} \tag{24}$$

where ∇ is the gradient operator. The computed reliability index β has a one-to-one nonlinear relationship with the failure probability P_f.

The HL–RF algorithm is currently the most widely used method for solving the constrained optimization problem in structural reliability.[12] The method is based on the following recursive formula:

$$U_{k+1} = \left[1\big/\nabla g^T(U_k)\nabla g(U_k)\right]\left[\nabla g^T(U_k)U_k - g(U_k)\right]\nabla g(U_k) \tag{25}$$

Experience has shown that for most situations, the HL–RF algorithm converges rapidly. It is the primary reliability methodology employed in the present study.

SORM involves the use of a quadratic approximation of the performance function, which, in the event of a truly nonlinear limit state surface, generally provides slightly improved results over those of FORM. Several algorithms have been developed for SORM, including the following.

Breitung[13]:

$$P_f = \Phi(-\beta)\prod_{j=1}^{n-1}(1 + \beta\kappa_j)^{-\frac{1}{2}} \tag{26}$$

Hohenbichler and Rackwitz[14]:

$$P_f = \Phi(-\beta)\prod_{j=1}^{n-1}\left(1+\frac{\phi(\beta)}{\Phi(\beta)}\kappa_j\right)^{\frac{1}{2}} \tag{27}$$

Tvedt[15]:

$$P_f = \Phi(-\beta)R_e\left[i\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_{t=0}^{\infty}\frac{1}{t}\right.$$

$$\left.\times\exp\left[\frac{(t+\beta)^2}{2}\right]\prod_{j=1}^{n-1}(1-t\kappa_j)^{-\frac{1}{2}}\,\mathrm{d}t\right] \tag{28}$$

In relations (26–28) β is the FORM-based reliability index, n is the number of random variables, and κ_j, $j = 1, 2, \ldots, n-1$, are the ordered main curvatures of the original failure surface in the u space at the design point. Alternatively, the Monte Carlo simulation technique, in which the failure set $g(X)$ is populated through generation of random samples, has proven a valuable instrument in reliability analysis. All of these techniques have been implemented in the COMPASS probabilistic analysis program.[16] The proposed methodology has been applied in the solution of several example problems using the FUZPROB fuzzy probabilistic analysis program, which uses COMPASS as its probabilistic computational engine.

Fuzzy Modeling Of Probabilistic Response

Many of the parameters included in the models used to describe fatigue crack growth in the presence of MSD are calibrated using a combination of experimental data and expert opinion, including the component width, thickness, applied stress range, material properties, loading, and correction factors. It is well accepted that no general consensus (even from SMEs) exists as to the best selection of deterministic values for these parameters, let alone the most appropriate probabilistic characteristics such as mean, standard deviation, correlation, and even distribution type. This selection is made even more difficult given that components are continuously subject to the aging process and knowledge regarding the impact of in-service aging is rather limited. Therefore, such parameters cannot be modeled accurately as purely deterministic or purely random variables and should be instead represented as fuzzy random variables, incorporating a subjective degree of fuzziness into their probabilistic characterization. Accordingly, a fuzzy modeling strategy must be employed to accommodate such parametric subjectivity.

Modeling of MSD-Related Fuzzy Random Input Variables

As already alluded to, some of the parameters used in fatigue crack laws and MSD models may be described as fuzzy random variables. More specifically, the probabilistic characteristics of these random variables (mean value, standard deviation, correlation, etc.) could be fuzzy to some degree because estimates are typically derived from both experimental data and elicitation of SMEs. The process by which fuzzy variables are quantified is known as fuzzification, which is accomplished by constructing a possibility distribution or membership function for the variable. An example of a skewed convex triangular possibility distribution for the mean value of a random variable is shown in Fig. 1, whereas a trapezoidal possibility distribution is shown in Fig. 2. The y axis reflects α, the level of knowledge regarding the mean value of the random variable, with values ranging from zero to one, where $\alpha = 0$ corresponds to little knowledge or highly uncertain and $\alpha = 1$ (referred to as the normal point) implies a high degree of knowledge or being very confident. The x axis suggests the upper and lower bounds between which the mean value of the input parameter is thought to lie for a given level of knowledge α. Similar representations can be carried out for the standard deviation of the fuzzy random variables. The fuzzification process involves an aggregation of expert opinion and experimental data and is not pursued in the current investigation. See Ross[17] for a more detailed description of the fuzzification process.

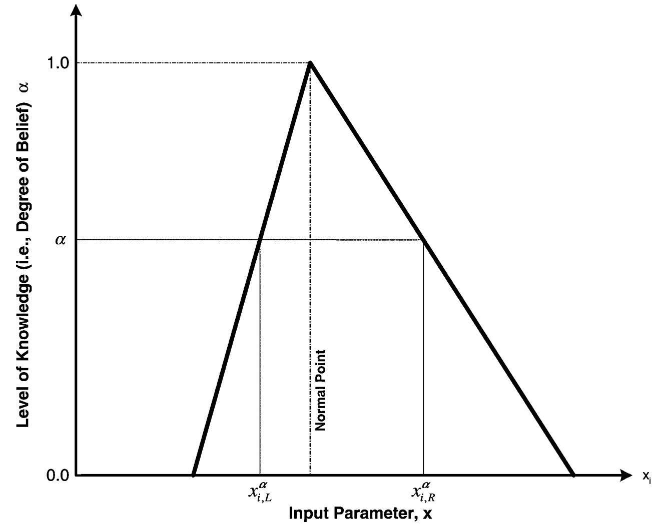

Fig. 1 Triangular possibility distribution of the mean value of a fuzzy Random Variable.

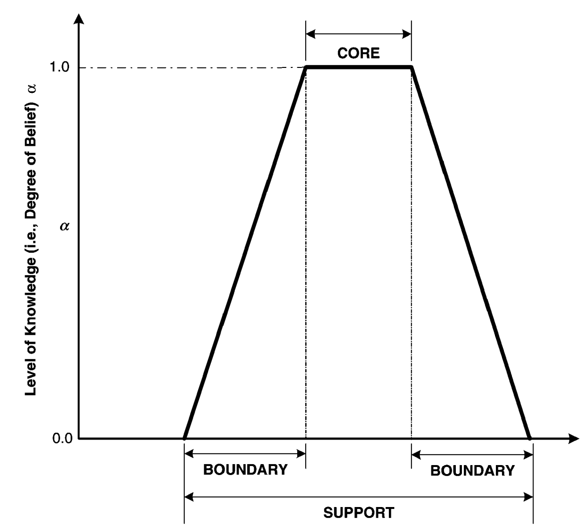

Fig. 2 Trapezoidal possibility distribution of the mean value of a fuzzy Random Variable.

Computation of Fuzzy Probabilistic Response

On fuzzification of the characteristic values of random variables, namely, mean, standard deviation, correlation, and probability distribution, the fuzzy probabilistic response quantity, that is, probability of failure or reliability index, must be computed, thereby implying that a possibility distribution also be constructed for the response quantity. Computation of fuzzy response is based on a fuzzy principle known as the extension principle, which, in some cases, may require an impractical amount of computational resources, depending on the number of fuzzy variables considered. A much-improved solution strategy, which circumvents the limitations of the extension principle, namely, the response surface method, was developed by Akpan et al.[18] The extension principle is briefly discussed in the following section, followed by a presentation of the response surface solution strategy.

Extension Principle

The extension principle associates the possibility distribution for the fuzzy input parameters with that of the fuzzy response function. Given a set of independent fuzzy variables x_i, $i = 1, 2, \ldots, N$, where N is the total number of fuzzy input parameters, on which a function $y(x_i)$ operates, the extension principle gives the possibility distribution μ_{xi} of the output function $c = y(x_i)$ (probability of failure or reliability index) as

$$\mu_{xi}[c = y(x_i)] = \sup|_c\Big[\min_i(\mu_{xi}\{y(x_i)\})\Big]$$

$$-\infty < y(x_i) < \infty, \qquad 1 < i < N \tag{29}$$

where sup represents the supernum operator that gives the least upper bound. Equation (29) essentially states that, for a crisp value of the output function $y(x_i)$, there exists either zero, one, or more combinations of the fuzzy variables x_i such that $y(x_i) = c$. In fuzzy set theory, the possibility of a particular outcome is equal to the minimum possibility of its constituent events. When several paths may be used to predict the outcome, each with its own possibility, then the overall possibility of this outcome is given by the maximum of all individual possibilities. Two approximate numerical methodologies, namely, the discretization method and the vertex method, are used to compute fuzzy parameters by means of conversion from fuzzy to crisp values. These techniques will be discussed in the following subsections.

Discretization method: The discretization method requires that each fuzzy input variable be discretized into a domain D_i of discrete values. The output function y must be evaluated a total of

$$\prod_{i=1}^{N}(D_i)$$

times. Consider, for example, a problem involving five fuzzy input parameters, each discretized into five values. In this case, the output function must be solved a total of $(5)^5 = 3125$ times. Unfortunately, this method requires a significant amount of computational resources, thereby, making it impractical for computing reliability indices and failure probabilities.

Vertex method: In terms of computational expense, the vertex method is much cheaper for numerical implementation of the extension principle. This technique is oftentimes referred to as the combinatorial method. The main steps required in the vertex or combinatorial method include 1) discretization of the fuzzy input parameters using an α-level representation; 2) collection of all binary combinations of the extreme left (L) and right (R), that is, lower and upper, respectively, values of all fuzzy variables at each α level; 3) computation of the fuzzy response function y for all binary combinations of fuzzy input variables; and finally 4) selection of the maximum and minimum values of the response function y at each α level.

As alluded to earlier, an α-level representation of a fuzzy variable X_i, denoted by the range $[X^{\alpha}_{i,L}, X^{\alpha}_{i,R}]$, indicates the interval in which the variable is thought to lie with a level of confidence equal to α. Therefore, at any α level, each fuzzy variable is discretized into a range of crisp values between $X^{\alpha}_{i,L}$ and $X^{\alpha}_{i,R}$. On discretization of the fuzzy input parameters, all combinations of their extreme L and R values (at a given α level) are fed into the output function y. Assuming a total of N fuzzy variables and denoting the combinations by $C_{\alpha,j}$, $j = 1, 2, \ldots, N_{c/\alpha}$, where $N_{c/\alpha}$ is the total number of binary combinations at a given α level, then the fuzzy response can be written as $y(C_{\alpha,j})$. It is easily shown that the total number of binary combinations at a given α level is equal to 2^N. The fuzzy response at each α level is given by

$$\left[y^{\alpha}_L, y^{\alpha}_R\right] = \{\min|_{\lambda,j}[y(C_{\lambda,j})], \qquad \max|_{\lambda,j}[y(C_{\lambda,j})]\}$$

$$\lambda \geq \alpha, \qquad j = 1, 2, \ldots, N_{C/\alpha} \tag{30}$$

For a range of α levels, the results of Eq. (30) are then used to construct a possibility distribution for the response function y. In general, the greater the number of α cuts, the more accurate is the predicted possibility distribution. For a number of α cuts A, the response function (probability of failure or reliability index) must be solved a total of $A \times 2^N$ times. It is evident that computational cost grows exponentially with increasing number of fuzzy input variables. For example, a problem involving 5 fuzzy variables and 5 α cuts requires that the objective function be called $5 \times 2^5 = 160$ times, whereas one involving 10 fuzzy parameters and 5 α cuts must call the output function a total of $5 \times 2^{10} = 5120$ times.

Response Surface Prediction of Fuzzy Response

The response surface method is a classical statistical technique wherein a complicated model is approximated using a simplified relationship between the fuzzy input variables and the fuzzy response function. Instead of collecting the binary combinations of upper and lower fuzzy inputs (mean values, standard deviations, correlation coefficients, etc., of fuzzy random variables), adjustments are made to the normal point of all fuzzy variables, and the resulting impact on fuzzy response is observed. Because most systems are very well behaved, the impact of such adjustments may be used to develop an explicit function, that is, response surface, relating the input and output variables. For a quadratic response surface, this function is given by

$$\bar{y}(x) = a + \sum_{i=1}^{N} b_i x_i + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} x_i x_j \tag{31}$$

where N is the number of fuzzy variables x, $\bar{y}(x)$ is the response function approximation, and the coefficients a, b, and c are determined through regression of experimental data. On calculation of the approximate fuzzy response function $\bar{y}(x)$, a combinatorial optimization is performed at each α level to determine the combination of fuzzy variables yielding the extreme values of fuzzy response. More specifically, two combinatorial optimizations are required at each α-level, with the following objectives:

1) Determine the combination of $x^{\alpha}_{i,L}$ and $x^{\alpha}_{i,R}$, denoted by $C_{\alpha/\min}$, that minimizes the approximate function $\bar{y}(x)$.

2) Determine the combination of $x^{\alpha}_{i,L}$ and $x^{\alpha}_{i,R}$, denoted by $C_{\alpha/\max}$, that maximizes $\bar{y}(x)$.

The true maximum and minimum responses at each α level are then determined by feeding the values of $C_{\alpha/\min}$ and $C_{\alpha/\max}$ into a finite element model or other numerical engine. This methodology requires that the response function be solved only twice at each α level. For a number of α cuts A, the response surface method requires only $A \times 2$ runs, whereas the vertex method requires $A \times 2^N$ runs. As noted by Akpan et al.,[18] the response surface method represents an enormous improvement in computational efficiency, without compromising predictive accuracy, and places no limit on the number of fuzzy variables that may be included. The response surface approach to fuzzy probability response computation is shown schematically in Fig. 3.

Example Demonstration

In the following section, an example will be considered to demonstrate the application of fuzzy probabilistic logic to the problem of multiple-site fatigue damage. A typical MSD scenario observed in a structural panel is shown in Fig. 4, in which a series of cracks is discovered emanating from a row of fastener holes.

For the current example, consider the fracture-based performance function for MSD, given previously by Eq. (17) as $g_{f,A_{\text{eff}}} = K_{IC} - (\psi_{\Delta K})\Delta K_{A_{\text{eff}}}$, where the mode 1 fracture toughness is given by K_{IC} and $\Psi_{\Delta K}$ is the correction factor for the bias in estimating the effective SIF range for the effective crack configuration, that is, $\Delta K_{A_{\text{eff}}}$. The effective SIF range was given earlier by Eq. (7) as $\Delta K_{\text{eff}} = \Delta\sigma\sqrt{\pi a}(Y_w Y_m)$. All model parameters are assumed to be either random variables or fuzzy random variables, with nominal probabilistic characteristics summarized in Table 1. The fuzzy input variables are assumed to be triangular in nature (Fig. 1). With emplayment the limit state function defined earlier, in conjunction with the response surface strategy outlined by Akpan et al.,[8,19] a fuzzy probabilistic analysis was performed. A total of 18 fuzzy variables were considered, namely, the mean and standard deviation of the nine MSD-related parameters summarized in Table 1. Uncertainty or fuzziness in the mean and standard deviation was assumed to lie within (+5%, −5%) for all parameters except fracture toughness K_{IC}, for which the fuzziness in probabilistic characteristics was assumed to lie within the skewed range of (+1%, −5%) to approximate the adverse effects of in-service aging.

The original probabilistic characteristics of each parameter were then used to obtain nominal FORM-based estimates of failure probability ($P_{f0} = 0.061$) and reliability index ($\beta_0 = 1.546$). The response surface technique outlined earlier was then applied to determine the possibility distributions for failure probability and reliability index. The possibility distribution for failure probability based on the use of fuzzy nominal means is shown in Fig. 5, whereas that based on the use of fuzzy nominal standard deviations is shown in Fig. 6. It is seen that the predicted bounds widen significantly as the level of knowledge decreases. It is interesting that, although the fuzziness in the input parameters varies linearly, the resulting mean-based possibility distribution exhibits a nonlinear variation, whereas that based on fuzzy standard deviations depicts linear behavior. The corresponding possibility distribution based on combined fuzzy input parameters, that is, both fuzzy means and fuzzy standard deviations, is shown in Fig. 7. For the current example, it is evident that uncertainty in the mean of the relevant MSD parameters has a much more pronounced effect on the possibility distribution for failure probability than does an equal degree of uncertainty in standard deviation.

Table 1 Probabilistic and fuzzy characterization of MSD random variables

Description of random variable	Distribution	Mean μ^a	Standard deviation σ^a
Applied stress range $\Delta\sigma$	Gumbel	650.000	100.0000
Crack size C a_C	Normal	0.010	0.0010
Crack size B a_B	Normal	0.010	0.0010
Crack size A a_A	Normal	0.006	0.0006
Tip separation between A and B b_{AB}	Lognormal	0.020	0.0020
Tip separation between B and C b_{BC}	Lognormal	0.020	0.0020
Fracture toughness[b] K_{IC}	Weibull	174.000	17.4000
Panel width W	Normal	0.400	0.0200
Bias correction factor Ψ	Normal	1.000	0.1000

[a]Fuzziness in μ and σ assumed to lie within ±5%.
[b]Fuzziness in K_{IC} assumed to lie within [+1%, −5%].

Figure 8 shows the possibility distribution for reliability index based on the use of fuzzy nominal means, whereas that based on the use of fuzzy nominal standard deviations is shown in Fig. 9. It is seen that the predicted bounds widen significantly as the level of knowledge decreases. Again, despite the linearly fuzzified input, the resulting fuzzy mean-based possibility distribution exhibits a nonlinear variation, whereas that based on fuzzy standard deviations depicts linear behavior. The corresponding possibility distribution based on combined fuzzy input parameters, that is, fuzzy means

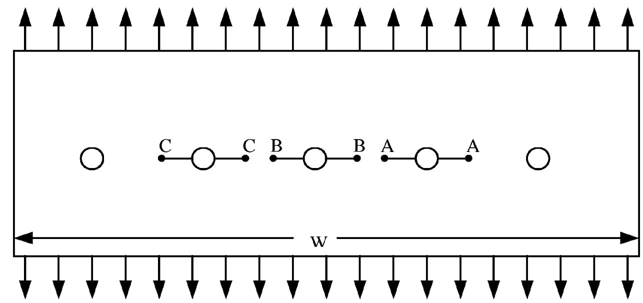

Fig. 4 Schematic of multi-site fatigue damage in an aging structure.

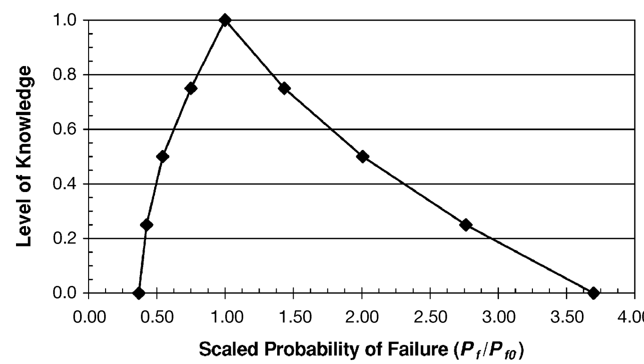

Fig. 5 Possibility distribution for probability of failure based on uncertainty in parametric mean, $P_{f0} = 0.061$.

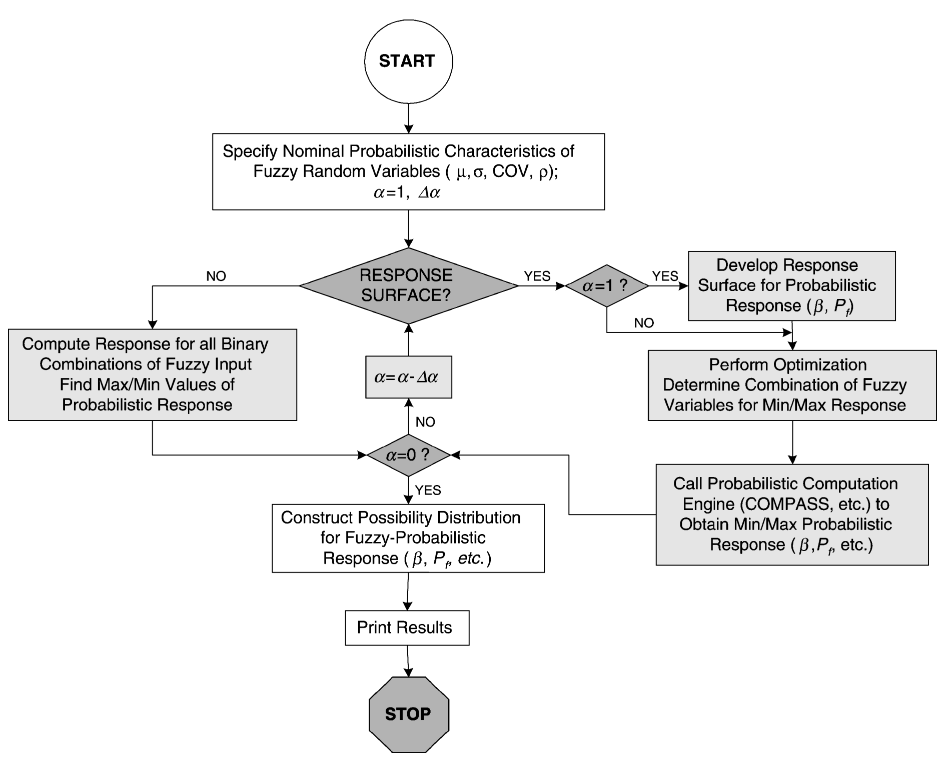

Fig. 3 Response surface strategy for fuzzy-probabilistic analysis of aging structures.

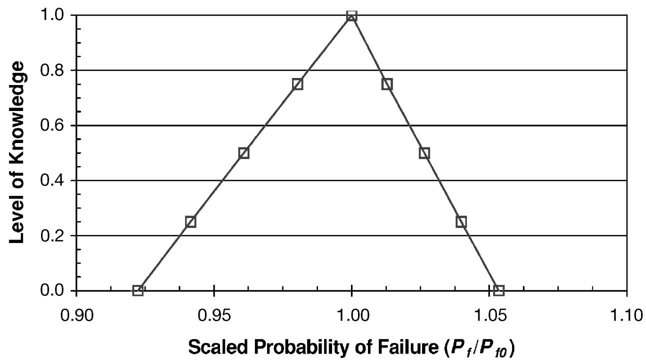

Fig. 6 Possibility distribution for probability of failure based on uncertainty in parametric standard deviation, $P_{f0} = 0.061$.

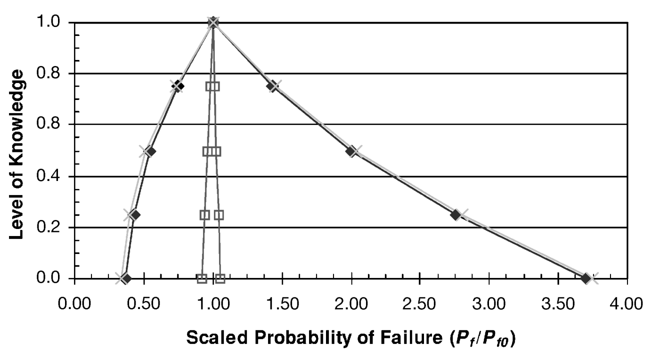

Fig. 7 Comparison of probability of failure-based possibility distributions for individual and combined fuzzy input parameter characteristics: ♦, mean; □, standard deviation; ×, combined mean and standard deviation.

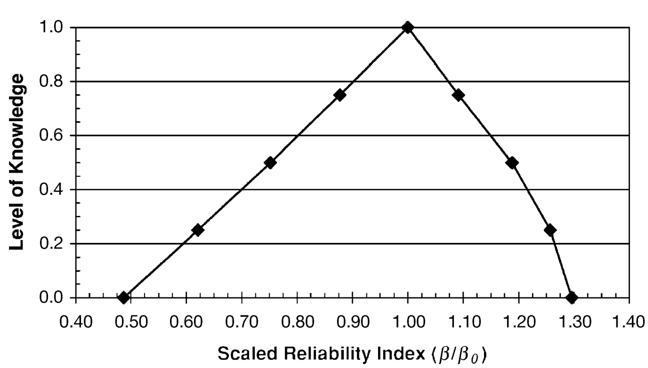

Fig. 8 Possibility distribution for reliability index based on uncertainty in parametric mean, $\beta_0 = 1.546$.

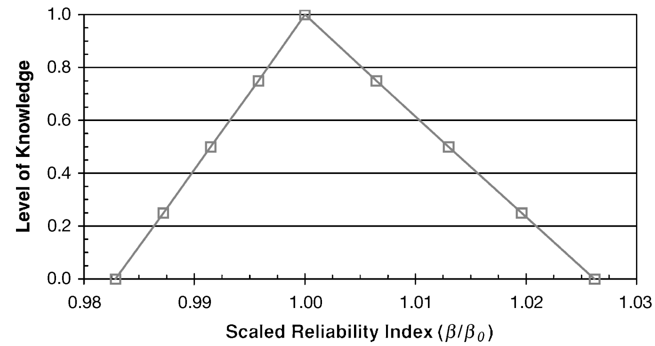

Fig. 9 Possibility distribution for reliability index based on uncertainty in parametric standard deviation, $\beta_0 = 1.546$.

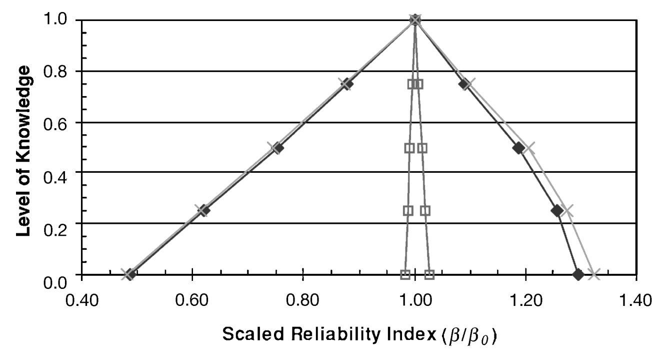

Fig. 10 Comparison of reliability index-based possibility distributions for individual and combined fuzzy input parameter characteristics: ♦, mean; □, standard deviation; ×, combined mean and standard deviation.

and standard deviations, is shown in Fig. 10. As was observed for probability of failure, it is evident that uncertainty in the mean of the relevant MSD parameters has a much more pronounced effect on the possibility distribution for reliability index than does an equal degree of uncertainty in standard deviation.

As already mentioned, in lieu of the traditional single-valued probabilistic results for β and P_f, the x axis of a fuzzy-based possibility distribution implies an upper and lower bound within which the objective, in this case reliability index and failure likelihood, is thought to lie for a given level of knowledge. For example, suppose there is 50% confidence in the fuzziness associated with both the mean and standard deviation of the nominal input parameters, that is, (+0.5%, −2.5%) for K_{IC} and (+2.5%, −2.5%) for all other input parameters. Figure 10 suggests that bounds on the corresponding reliability index would lie between 75% and 120% of its nominal value, thereby, making available to the decision maker both the worst- and best-case scenarios at this particular level of parametric fuzziness.

Recall that all fuzzy input parameters have linear unskewed triangular possibility distributions, that is, (+5%, −5%), with the exception of fracture toughness K_{IC}, which has a linearly skewed triangular possibility distribution, that is, (+1%, −5%). Note that the fuzzy input has resonated through the entire solution to produce nonlinearly skewed possibility distributions (Figs. 5–10). The predicted bounds on failure probability and reliability index based on the use of combined fuzzy random variables, that is, combined means and standard deviations, are wider than those predicted based on fuzzy nominal means and standard deviations considered individually. For the current example, use of the response surface technique (18 fuzzy variables and 5 α cuts) in lieu of the vertex method has reduced the number of required solutions from $5 \times 2^{18} = 1{,}310{,}720$ runs to only $5 \times 2 = 10$ runs, an reduction of more than five orders of magnitude.

Conclusions

MSD is a growing problem for aging structures (especially aircraft structures), one which is compounded by their continued operation well beyond their original design lives. Moreover, traditional policies for damage tolerance and structural integrity give little consideration to the potentially catastrophic repercussions associated with the neglect and nondetection of MSD and give even less consideration to the uncertainty inherent in many of the parameters involved.

Several mechanistic and probabilistic models for MSD have been presented. The potential for combining probabilistic analysis techniques with fuzzy modeling strategies has also been advanced, with focus on use of the response surface method.[8,19] Depending on the level of knowledge and degree of subjectivity, parameters affecting MSD may be represented as deterministic, random, fuzzy, or fuzzy random variables. A strategy for fuzzy probabilistic assessment of the impact of MSD on aging structures has been developed. FORM-based probabilistic solution strategies and response surface fuzzy modeling strategies have been combined to construct possibility distributions of the probabilistic response quantities, namely, reliability index β and failure probability P_f, for components subject to MSD. In lieu of providing a crisp value of structural reliability, the merit of the proposed methodology lies in its ability to implement a combination of experimental data and expert opinion in the prediction of confidence bounds on the structural integrity of aging structures. The predicted bounds are largely dependent on the level of knowledge regarding the probabilistic characteristics of the fuzzy input parameters, with a higher degree of knowledge producing much tighter bounds.

An example problem, which employs a fracture-based failure criterion for MSD, has been presented to demonstrate the proposed fuzzy probabilistic methodology. For this example problem, it was observed that, whereas reliability is greatly affected by uncertainty in parametric mean, variability in standard deviation is of much less significance. A similar investigation need not be limited to the uncertainties specified herein, but rather the degree of uncertainty for a given input parameter could be tailored to match observed variability in experimental data and expert opinion. Moreover, the current methodology may be employed in conjunction with any limit state function deemed appropriate for MSD (without modification to its basic formulation) and should not be limited to those mentioned herein. The input parameter fuzzification process has not been pursued during this work, but is discussed in detail in works by Ross[17] and Akpan et al.[18] To the best of the authors' knowledge, similar investigations have not been performed to date.

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